

# ALGEBRAIC APPROXIMATION PRESERVING DIMENSION

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**ABSTRACT.** We prove that each semialgebraic subset of  $\mathbb{R}^n$  of positive codimension can be locally approximated of any order by means of an algebraic set of the same dimension. As a consequence of previous results, algebraic approximation preserving dimension holds also for semianalytic sets.

## 1. INTRODUCTION

If  $A$  and  $B$  are two closed subanalytic subsets of  $\mathbb{R}^n$ , the Hausdorff distance between their intersections with the sphere of radius  $r$  centered at a common point  $P$  can be used to “measure” how near the two sets are at  $P$ . We say that  $A$  and  $B$  are  $s$ -equivalent (at  $P$ ) if the previous distance tends to 0 more rapidly than  $r^s$  (if so, we write  $A \sim_s B$ ).

In the papers [FFW1], [FFW2] and [FFW3] we addressed the question of the existence of an algebraic representative  $Y$  in the class of  $s$ -equivalence of a given subanalytic set  $A$  at a fixed point  $P$ . In this case we also say that  $Y$   $s$ -approximates  $A$ .

The answer to the previous question is in general negative for subanalytic sets (see [FFW2]).

On the other hand in [FFW1] it was proved that, for any real number  $s \geq 1$  and for any closed semialgebraic set  $A \subset \mathbb{R}^n$  of codimension  $\geq 1$ , there exists an algebraic subset  $Y$  of  $\mathbb{R}^n$  such that  $A \sim_s Y$ . The proof of the latter result consists in finding equations for  $Y$  starting from the polynomials appearing in a presentation of  $A$ . For instance if  $A = \{x \in \mathbb{R}^n \mid f(x) = 0, h(x) \geq 0\}$  with  $f, h \in \mathbb{R}[x]$ , then  $A$  can be  $s$ -approximated by the algebraic set  $Y = \{x \in \mathbb{R}^n \mid (f^2 - h^m)(x) = 0\}$  for any odd integer  $m$  sufficiently large. This procedure does not guarantee that  $Y$  has the same dimension as  $A$  at  $P$  as the following trivial example shows.

Let  $A$  be the positive  $x_3$ -axis in  $\mathbb{R}^3$  presented as  $A = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = 0, x_3 \geq 0\}$ . Then according to the previous procedure, for any sufficiently large odd integer  $m$ ,  $A$  is  $s$ -approximated at the origin  $O$  by the algebraic set  $Y = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_1^2 + x_2^2)^2 - x_3^m = 0\}$ , whose germ at  $O$  has dimension 2. However we can also  $s$ -approximate  $A$  at  $O$  by the 1-dimensional algebraic set  $W = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 - x_3^m = 0, x_2 = 0\}$  for any sufficiently large odd integer  $m$ . This algebraic set can be obtained by a similar construction as before but starting from the different presentation  $A = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = 0, x_2 = 0, x_3 \geq 0\}$ .

In [FFW3] we proved that, for any  $s \geq 1$ , any closed semianalytic subset  $A \subseteq \mathbb{R}^n$  is  $s$ -equivalent to a semialgebraic set  $Y \subset \mathbb{R}^n$  having the same local dimension as  $A$ . However the arguments used in the proof of this latter result do not guarantee that, even if  $A$  is analytic, it can be approximated by means of an algebraic one of the same dimension.

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In this paper we prove in Theorem 4.1 that any semialgebraic set of codimension  $\geq 1$  is  $s$ -equivalent to an algebraic one of the same dimension. Using the mentioned result of [FFW3], we obtain (Corollary 4.3) that any semianalytic set of codimension  $\geq 1$  can be  $s$ -approximated by an algebraic one preserving the local dimension. The proof of Theorem 4.1 works provided that the semialgebraic set is described by means of a suitable presentation, as in the previous example. Therefore Section 3 is devoted to introduce the notion of “regular presentation” and to prove that one can reduce to work with regularly presented sets.

## 2. BASIC PROPERTIES OF $s$ -EQUIVALENCE

In this section we recall the definition and some basic properties of  $s$ -equivalence of subanalytic sets at a common point which, without loss of genericity, we can assume to be the origin  $O$  of  $\mathbb{R}^n$ . We refer the reader to [FFW2] for the proofs of these results.

If  $A$  and  $B$  are non-empty compact subsets of  $\mathbb{R}^n$ , let  $\delta(A, B) = \sup_{x \in B} d(x, A)$ . Thus, denoting by  $D(A, B)$  the classical Hausdorff distance between the two sets, we have that  $D(A, B) = \max\{\delta(A, B), \delta(B, A)\}$ .

**Definition 2.1.** *Let  $A$  and  $B$  be closed subanalytic subsets of  $\mathbb{R}^n$  with  $O \in A \cap B$ . Let  $s$  be a real number  $\geq 1$ . Denote by  $S_r$  the sphere of radius  $r$  centered at the origin.*

(a) *We say that  $A \leq_s B$  if one of the following conditions holds:*

- (i)  *$O$  is isolated in  $A$ ,*
- (ii)  *$O$  is non-isolated both in  $A$  and in  $B$  and*

$$\lim_{r \rightarrow 0} \frac{\delta(B \cap S_r, A \cap S_r)}{r^s} = 0.$$

(b) *We say that  $A$  and  $B$  are  $s$ -equivalent (and we will write  $A \sim_s B$ ) if  $A \leq_s B$  and  $B \leq_s A$ .*

Observe that, if  $A \subseteq B$ , then  $A \leq_s B$  for any  $s \geq 1$ . It is easy to check that  $\leq_s$  is transitive and that  $\sim_s$  is an equivalence relation.

The following result shows the behavior of  $s$ -equivalence with respect to the union of sets:

**Proposition 2.2.** *Let  $A, A', B$  and  $B'$  be closed subanalytic subsets of  $\mathbb{R}^n$ .*

- (1) *If  $A \leq_s B$  and  $A' \leq_s B'$ , then  $A \cup A' \leq_s B \cup B'$ .*
- (2) *If  $A \sim_s B$  and  $A' \sim_s B'$ , then  $A \cup A' \sim_s B \cup B'$ .*

A useful tool to test the  $s$ -equivalence of two subanalytic sets is introduced in the following definition:

**Definition 2.3.** *Let  $A$  be a closed subanalytic subset of  $\mathbb{R}^n$ ,  $O \in A$ . For any real  $\sigma > 1$ , we will call horn-neighbourhood with center  $A$  and exponent  $\sigma$  the set*

$$\mathcal{H}(A, \sigma) = \{x \in \mathbb{R}^n \mid d(x, A) < \|x\|^\sigma\}.$$

**Remark 2.4.** If  $A$  is a closed semialgebraic subset of  $\mathbb{R}^n$  and  $\sigma$  is a rational number, then  $\mathcal{H}(A, \sigma)$  is semialgebraic. Moreover if  $O$  is isolated in  $A$ , then  $\mathcal{H}(A, \sigma)$  is empty near  $O$ . □

**Proposition 2.5.** *Let  $A, B$  be closed subanalytic subsets of  $\mathbb{R}^n$  with  $O \in A \cap B$  and let  $s \geq 1$ . Then  $A \leq_s B$  if and only if there exist real constants  $R > 0$  and  $\sigma > s$  such that*

$$(A \setminus \{O\}) \cap B(O, R) \subseteq \mathcal{H}(B, \sigma)$$

where  $B(O, R)$  denotes the open ball centered at  $O$  of radius  $R$ .

The following technical result shows that it is possible to modify a subanalytic set by means of a suitable horn-neighborhood producing a new subanalytic set  $s$ -equivalent to the original one:

**Lemma 2.6.** *Let  $X \subset Y \subset \mathbb{R}^n$  be closed subanalytic sets such that  $O \in X$  and let  $s \geq 1$ . Then:*

- (1) *for any  $\sigma > s$  we have  $Y \sim_s Y \cup \mathcal{H}(X, \sigma)$ ;*
- (2) *if  $\overline{Y \setminus X} = Y$ , there exists  $\sigma > s$  such that  $Y \setminus \mathcal{H}(X, \sigma) \sim_s Y$ .*

Another essential tool will be the following version of Lojasiewicz' inequality, proved in [FFW3]; henceforth for any map  $f: \mathbb{R}^n \rightarrow \mathbb{R}^p$  we will denote by  $V(f)$  the zero-set  $f^{-1}(O)$ .

**Proposition 2.7.** *Let  $A$  be a compact subanalytic subset of  $\mathbb{R}^n$ . Assume  $f$  and  $g$  are subanalytic functions defined on  $A$  such that  $f$  is continuous,  $V(f) \subseteq V(g)$ ,  $g$  is continuous at the points of  $V(g)$  and such that  $\sup |g| < 1$ . Then there exists a positive constant  $\alpha$  such that  $|g|^\alpha \leq |f|$  on  $A$  and  $|g|^\alpha < |f|$  on  $A \setminus V(f)$ .*

### 3. PRESENTATIONS OF SEMIALGEBRAIC SETS

This section is devoted to the first crucial step in our strategy, that is reducing ourselves to prove the main theorem for semialgebraic sets suitably presented.

**Definition 3.1.** *Let  $A$  be a closed semialgebraic subset of  $\mathbb{R}^n$  with  $\dim_O A = d > 0$ . We will say that  $A$  admits a good presentation if*

- (a) *the Zariski closure  $\overline{A}^Z$  of  $A$  is irreducible*
- (b) *there exist generators  $f_1, \dots, f_p$  of the ideal  $I(\overline{A}^Z) \subseteq \mathbb{R}[x_1, \dots, x_n]$  and  $h_1, \dots, h_q$  polynomial functions such that*

$$A = \{x \in \mathbb{R}^n \mid f_i(x) = 0, h_j(x) \geq 0, i = 1, \dots, p, j = 1, \dots, q\}$$

- (c)  *$h_i(O) = 0$  and  $\dim_O(V(h_i) \cap V(f)) < d$ , for each  $i$ , where  $f = (f_1, \dots, f_p)$ .*

**Lemma 3.2.** *Let  $A$  be a closed semialgebraic subset of  $\mathbb{R}^n$  with  $\dim_O A = d > 0$ . Then there exist closed semialgebraic sets  $\Gamma_1, \dots, \Gamma_r, \Gamma'$  such that*

- (1)  $A = (\bigcup_{i=1}^r \Gamma_i) \cup \Gamma'$
- (2) *for each  $i$ ,  $\dim_O \Gamma_i = d$ , and  $\dim_O \Gamma' < d$*
- (3) *for each  $i$ ,  $\Gamma_i$  admits a good presentation.*

*Proof.* Arguing as in [FFW3, Lemma 3.2] in the semialgebraic setting, there exist semialgebraic sets  $\Gamma_1, \dots, \Gamma_r, \Gamma'$  fulfilling conditions (1) and (2) of the thesis and such that, for each  $i$ ,  $\Gamma_i$  admits a presentation satisfying conditions (a) and (b) of Definition 3.1. In order to achieve also condition (c) it suffices to drop from the presentation of each  $\Gamma_i$  all the inequalities  $h_j(x) \geq 0$  such that  $h_j$  vanishes identically on  $\Gamma_i$ .  $\square$

Since we are interested in preserving dimension, we will reduce ourselves to work with a set presented by as many polynomial equations as its codimension and with the critical locus of the associated polynomial map nowhere dense.

**Notation 3.3.** Let  $\Omega$  be an open subset of  $\mathbb{R}^n$ . For any smooth  $\varphi: \Omega \rightarrow \mathbb{R}^p$ , denote  $\Sigma_r(\varphi) = \{x \in \Omega \mid \text{rk } d_x \varphi < r\}$  and  $\Sigma(\varphi) = \Sigma_p(\varphi)$ .

**Definition 3.4.** Let  $A$  be a closed semialgebraic subset of  $\mathbb{R}^n$  with  $\dim_O A = d > 0$ . We will say that  $A$  admits a regular presentation if there exist a polynomial map  $F: \mathbb{R}^n \rightarrow \mathbb{R}^{n-d}$  and polynomial functions  $h_1, \dots, h_q$  such that

- (a)  $A = \{x \in \mathbb{R}^n \mid F(x) = 0, h_j(x) \geq 0, j = 1, \dots, q\}$ ,
- (b)  $\dim_O(\Sigma(F) \cap A) < d$
- (c)  $h_i(O) = 0$  and  $\dim_O(V(h_i) \cap A) < d$ , for each  $i$ .

A useful tool to pass from a good presentation to a regular one will be the following result (for a proof see for instance [BCR, Proposition 7.7.10]):

**Lemma 3.5.** Let  $A$  be a closed semialgebraic subset of  $\mathbb{R}^n$  and let  $h, g$  polynomial functions on  $\mathbb{R}^n$ . Then there exist polynomial functions  $\varphi, \psi$  with  $\varphi > 0$  and  $\psi \geq 0$  such that

- (1)  $\text{sign}(\varphi h + \psi g) = \text{sign}(h)$  on  $A$
- (2)  $V(\psi) \subseteq \overline{V(h) \cap A}^Z$ .

**Proposition 3.6.** Let  $A$  be a closed semialgebraic subset of  $\mathbb{R}^n$  with  $\dim_O A = d > 0$  which admits a good presentation. Let  $s > 1$ . Then there exists a closed semialgebraic subset  $\tilde{A}$  of  $\mathbb{R}^n$  with  $\dim_O \tilde{A} = d > 0$  such that

- (1)  $\tilde{A}$  admits a regular presentation
- (2)  $\tilde{A} \sim_s A$ .

*Proof.* By hypothesis, we have that

$$A = \{x \in \mathbb{R}^n \mid f(x) = O, h_j(x) \geq 0, j = 1, \dots, q\}$$

with  $f = (f_1, \dots, f_p)$  such that  $V(f)$  is irreducible,  $V(f) = \overline{A}^Z$  and  $f_1, \dots, f_p$  generate the ideal  $I(V(f))$ . In particular  $\dim_O(\Sigma_{n-d}(f) \cap V(f)) < d$  (see for instance [BCR, Definition 3.3.3]).

If  $p = n - d$ , we have the thesis with  $\tilde{A} = A$ ; thus let  $p > n - d$ .

Denote by  $\Pi$  the set of surjective linear maps from  $\mathbb{R}^p$  to  $\mathbb{R}^{n-d}$  and consider the smooth map  $\Phi: (\mathbb{R}^n - V(f)) \times \Pi \rightarrow \mathbb{R}^{n-d}$  defined by  $\Phi(x, \pi) = (\pi \circ f)(x)$  for all  $x \in \mathbb{R}^n - V(f)$  and  $\pi \in \Pi$ .

The map  $\Phi$  is transverse to  $\{O\}$ : namely the partial Jacobian matrix of  $\Phi$  with respect to the variables in  $\Pi$  (considered as an open subset of  $\mathbb{R}^{p(n-d)}$ ) is the  $(n-d) \times p(n-d)$  matrix

$$\begin{bmatrix} f(x) & O & O & \dots & O \\ O & f(x) & O & \dots & O \\ \vdots & & & & \\ O & O & O & \dots & f(x) \end{bmatrix};$$

thus, for all  $x \in \mathbb{R}^n - V(f)$  and for all  $\pi \in \Pi$  the Jacobian matrix of  $\Phi$  has rank  $n - d$ .

As a consequence, by a well-known result of singularity theory (see for instance [BK, Lemma 3.2]), we have that the map  $\Phi_\pi: \mathbb{R}^n - V(f) \rightarrow \mathbb{R}^{n-d}$  defined by  $\Phi_\pi(x) = \Phi(x, \pi) = (\pi \circ f)(x)$  is transverse to  $\{O\}$  for all  $\pi$  outside a set  $\Gamma \subset \Pi$  of measure zero and hence  $\pi \circ f$  is a submersion on  $V(\pi \circ f) \setminus V(f)$  for all such  $\pi$ .

Let  $x \in V(f)$  be a point at which  $f$  has rank  $n - d$ . Then there is an open dense set  $U \subset \Pi$  such that for all  $\pi \in U$  the map  $\pi \circ f$  is a submersion at  $x$ , and hence off some subvariety of  $V(f)$  of dimension less than  $d$ .

Thus, if we choose  $\pi_0 \in (\Pi \setminus \Gamma) \cap U$ , the map  $F = \pi_0 \circ f: \mathbb{R}^n \rightarrow \mathbb{R}^{n-d}$  satisfies the following properties:

- $\dim_O V(F) = \dim_O V(f) = d$ ,
- $\Sigma(F) \cap V(F) \subseteq V(f) \subseteq V(F)$ ,
- $\dim_O(\Sigma(F) \cap V(F)) < d$ .

We want to show that there exist polynomials  $h'_i$  such that

- $A = \{x \in \mathbb{R}^n \mid f(x) = O, h'_i(x) \geq 0, i = 1, \dots, q\}$
- $\dim_O(V(F) \cap \bigcup_{i=1}^q V(h'_i)) < d$ .

Namely for each  $i \in \{1, \dots, q\}$  denote by  $W_i$  the union of the irreducible components  $Y$  of  $V(F)$  such that  $\dim_O(V(h_i) \cap Y) < d$ ; let also  $T_i = \overline{V(F) \setminus W_i}^Z$ . Note that  $V(f) \subseteq W_i$ .

If we apply Lemma 3.5 choosing  $h = h_i$  and  $g = \|f\|^2$  on  $W_i$ , then there exist  $\varphi, \psi$  with  $\varphi > 0$  and  $\psi \geq 0$  such that the function  $h'_i = \varphi h_i + \psi \|f\|^2$  has the same sign as  $h_i$  on  $W_i$  and  $V(\psi) \subseteq \overline{V(h_i) \cap W_i}^Z$ . Then

- $V(h'_i) \cap W_i = V(h_i) \cap W_i$
- since  $h'_i|_{T_i} = (\psi \|f\|^2)|_{T_i}$ , then  $V(h'_i) \cap T_i = (V(\psi) \cap T_i) \cup (V(f) \cap T_i) \subseteq W_i \cap T_i$ .

Thus  $\dim_O(V(h'_i) \cap V(F)) < d$  for any  $i$  and

$$A = \{x \in \mathbb{R}^n \mid f(x) = O, h'_i(x) \geq 0, i = 1, \dots, q\}.$$

For each  $m \in \mathbb{N}$  denote

$$(3.1) \quad \tilde{A}_m = \{x \in \mathbb{R}^n \mid F(x) = 0, \|x\|^{2m} - \|f(x)\|^2 \geq 0, h'_i(x) \geq 0, i = 1, \dots, q\}.$$

Since  $A \subseteq \tilde{A}_m \subseteq V(F)$ , then  $\dim_O \tilde{A}_m = d$ .

We claim that there exists  $m$  such that  $\tilde{A}_m \sim_s A$ . Since  $A \subseteq \tilde{A}_m$ , we trivially have that  $A \leq_s \tilde{A}_m$  for any  $m$ . Thus it is sufficient to prove that there exists  $m$  such that  $\tilde{A}_m \leq_s A$ . Namely, let  $\Lambda = \{x \in \mathbb{R}^n \mid h'_i(x) \geq 0, i = 1, \dots, q\}$ . Since  $V(\|f\|) \cap \Lambda = A = V(d(x, A)) \cap \Lambda$ , by Proposition 2.7 there exist a rational number  $\tau$  and a real number  $R > 0$  such that

$$d(x, A)^\tau < \|f(x)\| \quad \forall x \in (\Lambda \setminus V(f)) \cap B(O, R) = (\Lambda \setminus A) \cap B(O, R).$$

Let  $m > s\tau$ . Then  $d(x, A) < \|f(x)\|^{\frac{1}{\tau}} \leq \|x\|^{\frac{m}{\tau}}$  for all  $x \in (\tilde{A}_m \setminus A) \cap B(O, R)$ . This implies that  $(\tilde{A}_m \setminus \{O\}) \cap B(O, R) \subseteq \mathcal{H}(A, \frac{m}{\tau})$  and hence, by Proposition 2.5,  $\tilde{A}_m \leq_s A$ .

Up to increasing  $m$ , we can also assume that  $\dim_O(V(F) \cap V(\|x\|^{2m} - \|f(x)\|^2)) < d$  and hence that (3.1) is a regular presentation of  $\tilde{A}_m$ .

It is thus sufficient to choose  $m$  as above and  $\tilde{A} = \tilde{A}_m$ .

□

#### 4. MAIN RESULT

Since  $s$ -equivalence depends only on the germs at  $O$ , we are allowed to identify a subanalytic set with a realization of its germ at the origin in a suitable ball  $B(O, R)$ . Henceforth we will even omit to explicitly indicate the intersection of our sets with  $B(O, R)$ ; in particular, given two sets  $U$  and  $U'$ , when we write that  $U \subseteq U'$  we mean that  $U \cap B(O, R) \subseteq U' \cap B(O, R)$  for a suitable real constant  $R > 0$ .

**Theorem 4.1.** *For any real number  $s \geq 1$  and for any closed semialgebraic set  $A \subset \mathbb{R}^n$  of codimension  $\geq 1$  with  $O \in A$ , there exists an algebraic subset  $S$  of  $\mathbb{R}^n$  such that  $A \sim_s S$  and  $\dim_O S = \dim_O A$ .*

*Proof.* We will prove the thesis by induction on  $d = \dim_O A$ .

If  $d = 0$  the result holds trivially. So let  $d \geq 1$  and assume that the result holds for all semialgebraic sets of dimension less than  $d$ .

By Lemma 3.2, there exist closed semialgebraic sets  $\Gamma_1, \dots, \Gamma_r, \Gamma'$  such that

- (1)  $A = (\bigcup_{i=1}^r \Gamma_i) \cup \Gamma'$
- (2) for each  $i$ ,  $\dim_O \Gamma_i = d$  and  $\Gamma_i$  admits a good presentation
- (3)  $\dim_O \Gamma' < d$ .

By Proposition 2.2, by Proposition 3.6 and by the inductive hypothesis we can assume that  $A$  is described by means of a regular presentation as

$$A = \{x \in \mathbb{R}^n \mid F_0(x) = O, h_j(x) \geq 0, j = 1, \dots, q\}$$

with  $F_0 = (f_1, \dots, f_{n-d})$ . We can assume  $q \geq 1$ , because otherwise there is nothing to prove.

We will use the following notation:

- $Z_i = \bigcup_{j=i+1}^q V(h_j)$  for  $i = 0, \dots, q-1$ , and  $Z_q = \emptyset$ ,
- $X = (\Sigma(F_0) \cup Z_0) \cap A$ ,
- $\tilde{f} = (f_2, \dots, f_{n-d}): \mathbb{R}^n \rightarrow \mathbb{R}^{n-d-1}$  and  $V = V(\tilde{f})$ ,
- $\Lambda_i = \{x \in \mathbb{R}^n \mid h_j(x) \geq 0, j = i+1, \dots, q\}$  for any  $i = 0, \dots, q-1$ , and  $\Lambda_q = \mathbb{R}^n$ .

Since the presentation of  $A$  is regular, we have that

$$\dim_O(\Sigma(F_0) \cap A) < d \quad \text{and} \quad \dim_O(Z_0 \cap A) < d.$$

Let  $X_1 = X \cap \overline{A \setminus X}$  and  $X_2 = \overline{X \setminus X_1}$ . Since  $A = \overline{A \setminus X_1} \cup X_2$  and  $\dim_O X_2 < d$ , by the inductive hypothesis it suffices to prove the thesis for  $\overline{A \setminus X_1}$ .

In other words we can assume that  $A = \overline{A \setminus X}$ . As a consequence Lemma 2.6 shows that there exists a rational number  $\sigma > s$  such that, if  $K = \mathbb{R}^n \setminus \mathcal{H}(X, \sigma)$  then  $A \cap K \sim_s A$ .

Let  $g_0 = f_1$ . We will recursively construct polynomial functions  $g_1, \dots, g_q$  such that, if  $F_i = (g_i, f_2, \dots, f_{n-d})$ , then for any  $i = 0, \dots, q$  the semialgebraic subset

$$A_i = \{x \in \mathbb{R}^n \mid F_i(x) = 0, h_j(x) \geq 0, j = i+1, \dots, q\} = V(g_i) \cap V \cap \Lambda_i$$

satisfies the following properties

- P1(i):  $\begin{cases} A_i \sim_s A_{i-1} \text{ and } A_i \cap K \sim_s A_{i-1} \cap K & \text{if } i \in \{1, \dots, q\} \\ A_0 \cap K \sim_s A_0 & \text{if } i = 0 \end{cases}$
- P2(i):  $Z_i \cap A_i \cap K \subseteq \{O\}$
- P3(i):  $\Sigma(F_i) \cap A_i \cap K \subseteq \{O\}$ .

As proved above, the set  $A_0 = A$  satisfies the properties P1(0), P2(0) and P3(0). Thus assume that  $0 \leq i \leq q-1$ , assume that we have already constructed  $A_i$  fulfilling the three previous properties and let us construct  $g_{i+1}$  in such a way that  $A_{i+1}$  satisfies properties P1(i+1), P2(i+1) and P3(i+1).

For any positive integer  $m$  let  $g_{i+1} = g_i^2 - h_{i+1}^m$ .

We want to see that there exists  $m_s \in \mathbb{N}$  such that for any odd integer  $m \geq m_s$  the semialgebraic set  $A_{i+1} = V(g_{i+1}) \cap V \cap \Lambda_{i+2}$  satisfies properties P1(i+1), P2(i+1) and P3(i+1).

Properties P2(i) and P3(i) evidently guarantee that  $(A_i \cap K) \cap \Sigma(F_i) \cup Z_i \subseteq \{O\}$ . We will need to strengthen this fact as follows

**Claim:** There exists  $\beta > s$  such that  $\mathcal{H}(A_i \cap K, \beta) \cap (\Sigma(F_i) \cup Z_i) \subseteq \{O\}$ .

Namely, let  $\phi : \Sigma(F_i) \cup Z_i \rightarrow \mathbb{R}$  be the function defined by  $\phi(x) = d(x, A_i \cap K)$  for every  $x \in \Sigma(F_i) \cup Z_i$ . The function  $\phi$  is semialgebraic, continuous and, by the previous properties P2(i) and P3(i),  $V(\phi) = A_i \cap K \cap (\Sigma(F_i) \cup Z_i) \subseteq \{O\}$ . Hence by Proposition 2.7 there exists a rational positive number  $\beta$  such that  $d(x, A_i \cap K) > \|x\|^\beta$  for all  $x \in (\Sigma(F_i) \cup Z_i) \setminus \{O\}$ ; evidently we can assume that  $\beta > s$ . By definition of horn-neighborhood no  $x \neq O$  can lie in  $\mathcal{H}(A_i \cap K, \beta) \cap (\Sigma(F_i) \cup Z_i)$  which proves the Claim.

In particular for each  $j = i + 1, \dots, q$  we have that

$$(4.1) \quad h_j|_{B(x, \|x\|^\beta)} > 0 \quad \forall x \in A_i \cap K \setminus \{O\}.$$

Property P1(i+1). Consider the set  $E = \mathbb{R}^n \setminus \mathcal{H}(A_i \cap K, \beta)$ .

Evidently the closed semialgebraic set  $W = (V \cap \Lambda_{i+1} \cap K \cap E) \cap \{h_{i+1} \geq 0\}$  fulfills the condition

$$V(g_i) \cap W = (A_i \cap K) \cap E = \{O\}.$$

Thus by Proposition 2.7 there exists  $m_1 \in \mathbb{N}$  such that for any integer number  $m \geq m_1$  we have  $g_i(x)^2 \geq h_{i+1}(x)^m$  for all  $x \in W$  and  $g_i(x)^2 > h_{i+1}(x)^m$  for all  $x \in W \setminus \{O\}$ .

If we take  $m$  an odd integer  $\geq m_1$ , by construction  $g_{i+1} = g_i^2 - h_{i+1}^m$  is strictly positive on  $W \setminus \{O\}$  and on  $\{h_{i+1} < 0\}$ , hence  $g_{i+1}$  is strictly positive on  $(V \cap \Lambda_{i+1} \cap K \cap E) \setminus \{O\}$ . Since  $A_{i+1} = V(g_{i+1}) \cap V \cap \Lambda_{i+1}$ , it follows that

$$(4.2) \quad A_{i+1} \cap K \subseteq (\mathbb{R}^n \setminus E) \cup \{O\} = \mathcal{H}(A_i \cap K, \beta) \cup \{O\}$$

and therefore, by Proposition 2.5, that

$$A_{i+1} \cap K \leq_s A_i \cap K.$$

We want now to prove that  $A_i \cap K \leq_s A_{i+1} \cap K$ .

Consider the set  $B_i = V \cap \Lambda_i \supseteq A_i$ .

Assume at first that  $B_i \cap K \setminus \{O\}$  is connected and denote by  $d_g$  the geodesic distance on  $B_i \cap K$ ; denote also by  $B_g(x_0, r) = \{y \in B_i \cap K \mid d_g(y, x_0) < r\}$  the geodesic ball centered at  $x_0 \in B_i \cap K$ .

By [L1], up to working in a suitable Euclidean ball  $B(O, R)$ , there exist constants  $C > 0$  and  $0 < \alpha \leq 1$  such that for any  $y_1, y_2 \in B_i \cap K \cap B(O, R)$  we have that

$$\|y_1 - y_2\| \leq d_g(y_1, y_2) \leq C\|y_1 - y_2\|^\alpha.$$

Therefore we have

$$B_g(x_0, r) \subseteq B(x_0, r) \cap B_i \cap K \subseteq B_g(x_0, Cr^\alpha) \quad \forall x_0 \in B_i \cap K \cap B(O, R)$$

for  $r$  small enough. Up to decreasing  $R$  and  $\alpha$  if necessary, we can assume that  $C = 1$ .

Property P3(i) implies that, for any  $x \in A_i \cap K$ , we have that  $\dim_x(A_i \cap K) = d$  and, since  $\text{rk } d_x(\tilde{f}) = n - d - 1$ , that  $\dim_x(B_i \cap K) = d + 1$ . Hence  $\overline{(B_i \cap K) \setminus (A_i \cap K)} = B_i \cap K$ .

Thus Lemma 2.6 assures that there exists a closed semialgebraic subset  $K' \subseteq B_i \cap K$  such that

$$A_i \cap K' = A_i \cap K \cap K' = \{O\} \quad \text{and} \quad B_i \cap K \sim_{\frac{s+\beta}{\alpha}} K'.$$

Evidently  $V(g_i) \cap K' = V(g_i) \cap B_i \cap K' = A_i \cap K' = \{O\}$ . Thus by Proposition 2.7 there exists  $m_2 \in \mathbb{N}$  such that for any integer number  $m \geq m_2$  we have  $g_i(x)^2 \geq h_{i+1}(x)^m$  for all  $x \in K'$  and  $g_i(x)^2 > h_{i+1}(x)^m$  for all  $x \in K' \setminus \{O\}$ .

If we take  $m$  an integer  $\geq m_2$ , by construction  $g_{i+1} = g_i^2 - h_{i+1}^m$  is strictly positive on  $K' \setminus \{O\}$ .

Let  $x \in A_i \cap K \setminus \{O\}$ . As  $h_{i+1}(x) > 0$ , then  $g_{i+1}(x) < 0$ . Since  $B_i \cap K \sim_{\frac{s+\beta}{\alpha}} K'$ , there exist  $\eta > \frac{s+\beta}{\alpha}$  and  $z \in K'$  such that  $\|x - z\| < \|x\|^\eta$  (and we can assume that  $z \neq O$ ).

As  $g_{i+1}$  is strictly positive on  $K' \setminus \{O\}$ ,  $g_{i+1}(z) > 0$ . Since  $z \in B(x, \|x\|^\eta)$ , then  $z \in B_g(x, \|x\|^{\eta\alpha})$ . So, by the Intermediate Value Theorem on  $B_g(x, \|x\|^{\eta\alpha})$ , there exists  $w \in B_g(x, \|x\|^{\eta\alpha}) \subseteq B(x, \|x\|^{\eta\alpha}) \cap B_i \cap K$  such that  $g_{i+1}(w) = 0$ .

Moreover, as  $\eta\alpha > \beta$ , by (4.1) one has in particular that  $h_j(w) > 0$  for any  $j = i+2, \dots, q$ , which means that  $w \in A_{i+1} \cap K$ ; hence  $x \in \mathcal{H}(A_{i+1} \cap K, \eta\alpha)$ .

We have thus proved that  $A_i \cap K \setminus \{O\} \subseteq \mathcal{H}(A_{i+1} \cap K, \eta\alpha)$  and therefore, since  $\eta\alpha > s$ , that

$$A_i \cap K \leq_s A_{i+1} \cap K$$

by Proposition 2.5.

In the general case, if  $B_i \cap K \setminus \{O\}$  is not connected, it is sufficient to perform the previous argument on each connected component  $\Delta$  of  $B_i \cap K \setminus \{O\}$ , find an odd integer number  $m_\Delta$  as above, and take  $m_2 = \max m_\Delta$ .

Hence, if we let  $M = \max\{m_1, m_2\}$ , then for any odd  $m \geq M$  we have

$$(4.3) \quad A_{i+1} \cap K \sim_s A_i \cap K.$$

In order to conclude the proof that  $A_{i+1}$  satisfies property P1(i+1) observe that evidently

$$(4.4) \quad A_{i+1} \supseteq A_{i+1} \cap K \sim_s A_i \cap K \sim_s A \cap K \sim_s A$$

and thus in particular  $A_{i+1} \geq_s A$ .

For any  $\sigma'$  with  $\sigma > \sigma' > s$  we have that  $(A_{i+1} \cap \overline{\mathcal{H}(X, \sigma)}) \setminus \{O\} \subseteq \mathcal{H}(X, \sigma')$ ; hence  $A_{i+1} \cap \overline{\mathcal{H}(X, \sigma)} \leq_s X \subseteq A$  and thus  $A_{i+1} \cap \overline{\mathcal{H}(X, \sigma)} \leq_s A$ . Moreover from (4.4)  $A_{i+1} \cap K \leq_s A$ . Since

$$A_{i+1} = (A_{i+1} \cap K) \cup (A_{i+1} \cap \overline{\mathcal{H}(X, \sigma)}),$$

by Proposition 2.2 we have  $A_{i+1} \leq_s A$  and thus  $A_{i+1} \sim_s A$ . By the inductive hypothesis we also get that

$$A_{i+1} \sim_s A_i$$

and so P1(i+1) is proved.

Property P2(i+1). By (4.2) and the previous Claim, we have that

$$(4.5) \quad A_{i+1} \cap K \cap (\Sigma(F_i) \cup Z_i) \subseteq \{O\}.$$

Thus  $h_j$  does not vanish on  $A_{i+1} \cap K \setminus \{O\}$  for any  $j \geq i+1$ , which in particular proves that  $A_{i+1}$  satisfies property P2(i+1).



Property P3(i+1). In order to prove P3(i+1) consider the Jacobian matrix of  $F_{i+1} = (g_{i+1}, f_2, \dots, f_{n-d})$ , i.e.

$$\begin{pmatrix} 2g_i \nabla g_i - m h_{i+1}^{m-1} \nabla h_{i+1} \\ \nabla f_2 \\ \vdots \\ \nabla f_{n-d} \end{pmatrix}.$$

Evaluating it on the points of  $A_{i+1}$  we get the matrix

$$\begin{pmatrix} h_{i+1}^{\frac{m}{2}} (2\nabla g_i - m h_{i+1}^{\frac{m}{2}-1} \nabla h_{i+1}) \\ \nabla f_2 \\ \vdots \\ \nabla f_{n-d} \end{pmatrix}.$$

Since, as seen above,  $h_{i+1}$  does not vanish on  $A_{i+1} \cap K \setminus \{O\}$ ,

$$\Sigma(F_{i+1}) \cap A_{i+1} \cap K = \{x \in A_{i+1} \cap K \mid (2\nabla g_i - m h_{i+1}^{\frac{m}{2}-1} \nabla h_{i+1}) \wedge \nabla f_2 \wedge \dots \wedge \nabla f_{n-d} = 0\}.$$

If we let  $\varphi = 4\|\nabla g_i \wedge \nabla f_2 \wedge \dots \wedge \nabla f_{n-d}\|^2$  and  $\psi = m^2\|\nabla h_{i+1} \wedge \nabla f_2 \wedge \dots \wedge \nabla f_{n-d}\|^2$  we have that

$$\Sigma(F_{i+1}) \cap A_{i+1} \cap K = \{x \in A_{i+1} \cap K \mid \varphi(x) = |h_{i+1}(x)|^{m-2} \psi(x)\}.$$

Since  $V(\varphi) = \Sigma(F_i)$  and by (4.5)  $V(\varphi) \cap A_{i+1} \cap K \subseteq \{O\}$ , by Proposition 2.7 there exists  $\lambda$  such that  $\varphi(x) \geq \|x\|^\lambda$  on  $A_{i+1} \cap K$ . For the same reason there exists  $\mu$  such that  $|h_{i+1}(x)|^\mu \leq \|x\|$  on  $A_{i+1} \cap K$ . Moreover there exists a constant  $N$  such that  $\psi \leq N$  on  $A_{i+1} \cap K$ .

If  $m > \lambda\mu + 2$ , then  $\Sigma(F_{i+1}) \cap A_{i+1} \cap K \subseteq \{O\}$ . Namely, if by contradiction there exists a sequence of points  $x_\nu \in A_{i+1} \cap K$  converging to  $O$  such that  $\varphi(x_\nu) = |h_{i+1}(x_\nu)|^{m-2} \psi(x_\nu)$ , then

$$\|x_\nu\|^{\lambda\mu} \leq N^\mu \|x_\nu\|^{m-2}$$

which is a contradiction.

Let  $m_3$  be an integer such that  $m_3 > \lambda\mu + 2$ . Thus for any odd integer  $m \geq m_3$  we have that  $A_{i+1}$  satisfies property P3(i+1). In particular  $\dim_O(A_{i+1} \cap K) = d$ .

Finally, if we let  $m_s = \max\{M, m_3\}$ , then for any odd integer  $m \geq m_s$  we have that  $A_{i+1}$  satisfies all the properties P1(i+1), P2(i+1) and P3(i+1).

At the end of the recursive construction, observe that the set  $A_q$  is algebraic,  $A_q \sim_s A$ ,  $A_q \cap K \sim_s A \cap K \sim_s A$  and  $\dim_O(A_q \cap K) = d$ . Moreover

$$\overline{A_q \cap K}^Z \leq_s A_q \leq_s A \leq_s A_q \cap K \leq_s \overline{A_q \cap K}^Z.$$

Thus  $S = \overline{A_q \cap K}^Z$  satisfies the thesis.  $\square$

The previous theorem allows us to strengthen the following result on approximation preserving dimension which can be found in [FFW3]:

**Theorem 4.2.** *Let  $A$  be a closed semianalytic subset of  $\mathbb{R}^n$  with  $O \in A$ . Then for any  $s \geq 1$  there exists a closed semialgebraic set  $S \subseteq \mathbb{R}^n$  such that  $A \sim_s S$  and  $\dim_O S = \dim_O A$ .*

From Theorem 4.1 and from Theorem 4.2 we immediately obtain:

**Corollary 4.3.** *For any real number  $s \geq 1$  and for any closed semianalytic set  $A \subset \mathbb{R}^n$  of codimension  $\geq 1$  with  $O \in A$ , there exists an algebraic subset  $S$  of  $\mathbb{R}^n$  such that  $A \sim_s S$  and  $\dim_O S = \dim_O A$ .*

**Example 4.4.** If  $A = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0, x \geq 0, y \geq 0\}$  and  $s \geq 1$ , the approximation technique described in the proof of Theorem 4.1 yields a surface defined by  $(z^2 - x^m)^2 - y^p = 0$  for suitable odd integers  $m$  and  $p$ ; the shape of such a surface is represented in Figure 1

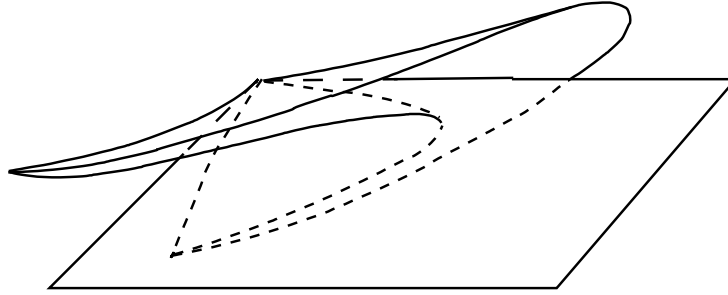


FIGURE 1. Algebraic approximation of a quadrant

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